On billiard weak solutions of nonlinear PDE's and Toda flows *

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Abstract

A certain class of partial differential equations possesses singular solutions having discontinuous first derivatives ("peakons"). The time evolution of peaks of such solutions is governed by a finite dimensional completely integrable system. Explicit solutions of this system are constructed by using algebraic-geometric method which casts it as a flow on an appropriate Riemann surface and reduces it to a classical Jacobi inversion problem. The algebraic structure of the finite dimensional flow is also examined in the context of the Toda flow hierarchy. Generalized peakon systems are obtained for any simple Lie algebra and their complete integrability is demonstrated.

1 Introduction

Camassa and Holm [1993] introduced and studied classes of soliton-type weak solutions, for an integrable nonlinear equation derived in the context of a shallow water model. In particular, they described the soliton dynamics in terms of a system of Hamiltonian equations for the locations of the "peaks" of the solution, the points at which its spatial derivative changes sign. In other words, each peaked solution, or peakon, can be associated with a mechanical system of moving particles. New systems of this type were obtained in Calogero [1995] and Calogero and Francoise [1996]. The r-matrix approach was applied to the Lax pair formulation of an n-peakon system by Ragnisco and Bruschi [1996], who also pointed out the connection of this system with the classical Toda lattice. A discrete version of the Adler-Kostant-Symes factorization method was used by Suris [1996] to study a discretization of the peakons lattice, realized as a discrete integrable system on a certain Poisson submanifold of gl(n) equipped with r-matrix Poisson bracket.

In Alber *et al.* [1994, 1997, 1999] the existence of peakons was linked to the presence of poles in the energy dependent Schrődinger operators associated with integrable evolution equations. Namely, it was shown that the presence of a pole in the potential is essential for a special limiting procedure which allows for the formation of "billiard' weak solutions. By using the algebraic-geometric method, these billiard solutions are related to finite dimensional integrable dynamical systems with reflections. In this way, profiles of billiard weak solutions are associated with billiard motion inside quadrics with and without the presence of a Hooke's potential. The points of impact on the quadrics correspond to the peaks of the profiles of weak solutions of the nonlinear PDE. Billiard solutions include new quasi-periodic and soliton-like solutions, as well as peaked solitons with compact support. This method can be used for a number of equations including the shallow water equation, the Dym type equation, as well as N-component systems with poles, together with all the equations in their respective hierarchies.

In Section 2 of this paper we derive equations for the motion of the peaks of billiard solutions in the context of the algebraic-geometric approach. We construct solutions of these equations by connecting them to a flow on an appropriate Riemann surface which leads to a classical Jacobi inversion problem.

In Section 3 we show that orbits of the Toda flow in $sl(n)$ into which the peakons lattice is mapped by means of the Lax matrix introduced in Ragnisco and Bruschi [1996] admits a natural generalization to the case of an arbitrary Lie algebra $\mathfrak{g}$. Restrictions of the Toda flow in $\mathfrak{g}$ to these orbits can be viewed as generalized peakons lattices. Using the methods developed in Gekhtman and Shapiro [1999] we give an intrinsic description of these orbits and prove complete integrability of the corresponding flows. Unlike in the $sl(n)$ case, Chevalley invariants of $\mathfrak{g}$ are not sufficient to ensure integrability in the general situation, and so we produce the requisite number of additional first integrals. Lastly, we explain how to construct Darboux coordinates for generalized peakon orbits and consider $sl(n)$ and G_2 as examples.

2 The Dynamics of the Peak Points

As shown in Alber *et al.* [1994, 1999], quasi-periodic solutions of the shallow water equation

$$U_t + 3UU_x = U_{xxt} + 2U_xU_{xx} + UU_{xxx}\,, \tag{2.1}$$

on the infinite line can be represented in the form

$$U(x,t) = \mu_1 + \cdots + \mu_n - \mathfrak{m} \tag{2.2}$$

where μ's are solutions of the following systems of equations in x and t,

$$\mu_i' = \frac{\partial \mu_i}{\partial x} = \mathrm{Sign}(\mu_i, m_{2i-1}, m_{2i}, s_i)\frac{\sqrt{R(\mu_i)}}{\mu_i \prod_{j\neq i}^{n}(\mu_i - \mu_j)} \tag{2.3}$$

and

$$\dot{\mu}_i = \frac{\partial \mu_i}{\partial t} = \mathrm{Sign}(\mu_i, m_{2i-1}, m_{2i}, d_i)\frac{(\mu_i - \Sigma)\sqrt{R(\mu_i)}}{\mu_i \prod_{j\neq i}^{n}(\mu_i - \mu_j)},$$
$$i = 1, \ldots, n, \quad \Sigma = \mu_1 + \cdots + \mu_n, \tag{2.4}$$

with

$$R(\mu) = \mu \prod_{r=1}^{2n+1} (\mu - m_r)\,. \tag{2.5}$$

Here the constants m_r depend on the initial condition for the equation (2.1), $\mathfrak{m} = \sum_r^{2n+1} m_r$, and $\mathrm{Sign}(y, l, r, s)$ is a function on the Riemann surface, $\Gamma = \{w^2 = R(\mu)\}$, which switches from -1 to 1 and back each time y reaches l or r, endpoints of a cut on the Riemann surface. These ordinary differential equations (ODE's) for the μ-variables provide half of the equations of a finite dimensional Hamiltonian system. Notice that periodic solutions form a subclass of the quasi-periodic solutions. Also, soliton solutions can be obtained from quasi-periodic solutions by shrinking pair wize the roots m_r of the spectral polynomial $R(E)$.

Peaked quasi-periodic solutions of equation (2.1) can be constructed from solutions of (2.3) and (2.4) by using the limiting process $m_1 \to 0$ (see Alber *et al.* [1994,1999]) and the trace formula (2.2). In what follows ODE's governing the time evolution of the peak locations are obtained and a connection with the canonical variables of Camassa and Holm [1993] is described.

The x and t evolution of the μ variables can also be connected to a constrained motion of a particle on an n-dimensional hypersurface imbedded in $\mathbb{R}^{n+1}$ which is parameterized by the constants m_r, under the action of a harmonic force field (see Alber *et al.* [1999]). The limit $m_1 \to 0$ corresponds to "flattening" this surface along one direction which results in harmonically forced motion in a region of $\mathbb{R}^n$ confined by an $(n-1)$-dimensional boundary. The particle collides with the boundary and moves along segments of an n-dimensional billiard. The reflection on the boundary causes, after using the trace formula (2.2), the appearance of peaks in the corresponding PDE solutions of equation (2.1).

On the level of (2.3) and (2.4), the boundary of the billiard can be shown to be described by the following condition: $\mu_1(x,t) = 0$. Reflection on the boundary is described by the sign switch from 1 to -1 (or from -1 to 1) of first derivative of μ_1 in (2.3) and (2.4).

The Periodic Case. The main steps in deriving equations for the peak locations can be demonstrated by considering the following basic example. Let us assume $n = 2$, and $m_1 = 0$. Equation $\mu_1(x,t) = 0$ defines a function of time $x = q(t)$. An equation for the time dependence of $q(t)$ can be found by differentiating $(\mu_1(q(t),t) = 0)$ with respect to time. We have

$$\frac{d}{dt}\Big(\mu_1(q(t),t)\Big) = 0 = \frac{\partial\mu_1}{\partial x}(q(t),t)\dot{q} + \frac{\partial\mu_1}{\partial t}(q(t),t). \tag{2.6}$$

The commuting x- and t-flows of μ_1 are described by

$$\begin{aligned}\frac{\partial\mu_1}{\partial x} &= \mathrm{Sign}(\mu_1)\frac{\sqrt{C_4(\mu_1)}}{\mu_1-\mu_2}\,, \\ \frac{\partial\mu_1}{\partial t} &= \mathrm{Sign}(\mu_1)\frac{(-\mu_2)\sqrt{C_4(\mu_1)}}{\mu_1-\mu_2}, \\ C_4(\mu) = (\mu - m_2)&(\mu - m_3)(\mu - m_4)(\mu - m_5).\end{aligned} \tag{2.7}$$

Notice that the spatial derivative of μ_1 is not defined at $x = q(t)$, so that in the above formula $\partial\mu_1/\partial x$ evaluated at $(q(t),t)$ needs to be found in the limit as $x \to q$ either from the left, or from the right. The sign uncertainty connected with this choice factors out from the final formula for $\dot{q}$. This combined with (2.6) and the billiard condition at the boundary $\mu_1(q(t),t) = 0$ gives

$$\dot{q} = \mu_2(q(t),t)\,. \tag{2.8}$$

We also need an evolution equation for the quantity $\mu_2(q(t),t)$. Notice that from the trace formula (2.2) and the expression for the boundary $\mu_1(q(t),t) = 0$, this is the quantity that determines the amplitude of the peak $U(q(t),t)$. The time derivative of $\mu_2(q(t),t)$ is

$$\frac{d}{dt}\Big(\mu_2(q(t),t)\Big) = \frac{\partial\mu_2}{\partial x}(q(t),t)\dot{q} + \frac{\partial\mu_2}{\partial t}(q(t),t)\,, \tag{2.9}$$

which used with the commuting x- and t-flows equations for μ_2 yields, in analogy with system (2.8),

$$\begin{aligned}\frac{dy}{dt}(t) &= -\,\mathrm{Sign}(y)\frac{\sqrt{C_4(y)}}{y}\Big(\dot{q} + B_1(y)\Big) \\ &= -\,\mathrm{Sign}(y)\frac{\sqrt{C_4(y)}}{y}\dot{q}\,.\end{aligned} \tag{2.10}$$

Here we have introduced the notation

$$y(t) = \mu_2(q(t),t)$$

and used again the definition of $q(t)$ so that

$$B_1(\mu_2(q(t)) \equiv B_1(y) = -\mu_1(q(t),t) = 0.$$

Combining this equation for $\dot{y}$ with the evolution equation (2.8) for q gives the following system for the variables y and q,

$$\begin{aligned}\dot{y} &= -\,\mathrm{Sign}(y)\sqrt{(y-m_2)(y-m_3)(y-m_4)(y-m_5)} \\ \dot{q} &= y\end{aligned} \tag{2.11}$$

Notice that the evolution for y is decoupled from that of q. The inversion problem corresponding to the equation for y determines a periodic function of t with nonzero average. The evolution of q is then given by the combination of a linear growth in time with slope given by the average of y, plus periodic oscillations. We also remark that for weak solutions of equation (2.1) having isolated discontinuities in the first derivative, the jump condition yields

$$\frac{dq}{dt} = U(q(t), t),$$

and using the trace formula (2.2) this can be seen to coincide with the second equation in system (2.11).

The Genus 2 Quasi-periodic Case. We describe a billiard genus 2 quasi-periodic solution. The spectral polynomial in this case is of the 5th order and will be denoted as $C_5(\mu)$. The boundary is again described by $\mu_1(q(t), t) = 0$. We shall write

$$y_1 = \mu_2(q(t), t) \quad \text{and} \quad y_2 = \mu_3(q(t), t).$$

From the weak solution approach it follows that $\dot{q} = U(q_1) = y_1 + y_2$. We differentiate y_1 and y_2 to obtain

$$\left.\begin{aligned} \dot{y}_1 &= \frac{d}{dt}\left[\mu_2(q,t)\right] = \frac{\partial \mu_2}{\partial x}\dot{q} + \frac{\partial \mu_2}{\partial t} = \frac{\operatorname{Sign}(\mu_2)\sqrt{C_6(\mu_2)}}{\mu_2(\mu_2 - \mu_3)}(\dot{q} - \mu_3) = \frac{\operatorname{Sign}(y_1)\sqrt{C_6(y_1)}}{y_1 - y_2} \\ \dot{y}_2 &= \frac{d}{dt}\left[\mu_3(q,t)\right] = \frac{\partial \mu_3}{\partial x}\dot{q} + \frac{\partial \mu_3}{\partial t} = \frac{\operatorname{Sign}(\mu_3)\sqrt{C_6(\mu_3)}}{\mu_3(\mu_3 - \mu_2)}(\dot{q} - \mu_2) = \frac{\operatorname{Sign}(y_2)\sqrt{C_6(y_2)}}{y_2 - y_1}. \end{aligned}\right\} \quad (2.12)$$

This is a well-defined system associated with a genus 2 Riemann surface. The corresponding problem of inversion involves only holomorphic differentials and therefore the quantities y_1,y_2 and $q(t)$ can be expressed in terms of standard θ-functions on the Riemann surface.

Peakon solutions. We now specialize the above formalism to the limiting case of soliton solutions of equation (2.1) on the real line, in which each pair m_{2i}, m_{2i+1} is taken to limit to the a_i, $i = 1, \ldots, n$. For just one μ-variable, using the trace formula $U = \mu - a$ at the billiard boundary results in $U(q(t), t) = -a$. The single peakon solution determined by this procedure is therefore

$$U(x,t) = -a e^{-|x+at|}\,, \quad (2.13)$$

which is a traveling wave soliton-type solution. Camassa and Holm [1993] found

$$U(x,t) = p(t) e^{-|x-q(t)|}\,, \quad (2.14)$$

with an additional link between $p(t)$ and $q(t)$ (Hamiltonian structure) which in the 1-peakon case results in $p(t)$ being a constant and $q(t)$ being a linear function.

2-peakon solution. In this case, one obtains the following system that describes 2-peakon profiles,

$$\left.\begin{aligned} \frac{\partial \mu_1}{\partial x} &= \operatorname{Sign}(\mu_1)\frac{(\mu_1 - a_1)(\mu_1 - a_2)}{(\mu_1 - \mu_2)} \\ \frac{\partial \mu_2}{\partial x} &= \operatorname{Sign}(\mu_2)\frac{(\mu_2 - a_1)(\mu_2 - a_2)}{(\mu_2 - \mu_1)} \end{aligned}\right\} \quad (2.15)$$

where μ_1 and μ_2 are evaluated between a_1 and 0 and a_2 and 0, respectively. The corresponding time-flow is

$$\left.\begin{aligned}\frac{\partial \mu_1}{\partial t} &= \mathrm{Sign}(\mu_1) B_1(\mu_1) \frac{(\mu_1 - a_1)(\mu_1 - a_2)}{(\mu_1 - \mu_2)} \\ \frac{\partial \mu_2}{\partial t} &= \mathrm{Sign}(\mu_2) B_1(\mu_2) \frac{(\mu_2 - a_1)(\mu_2 - a_2)}{(\mu_2 - \mu_1)}\end{aligned}\right\} \tag{2.16}$$

and the first order polynomial B_1 for equation (2.1) gives

$$B_1(\mu_1) = -\mu_2 + a_1 + a_2, \qquad B_1(\mu_2) = -\mu_1 + a_1 + a_2 \,.$$

Define the boundary by introducing functions $q_1(t)$ and $q_2(t)$ such that

$$\mu_1(q_1(t), t) = 0, \ \ \mu_2(q_2(t), t) = 0 \,. \tag{2.17}$$

Define functions

$$y_1 = \mu_2(q_1(t), t), \ \ y_2 = \mu_1(q_2(t), t). \tag{2.18}$$

Differentiating $\mu_1(q_1(t), t)$ and $\mu_2(q_2(t), t)$ in (2.17) results in

$$\left.\begin{aligned}\frac{d\mu_1}{dt} = 0 = \frac{\partial \mu_1}{\partial x}\left(\frac{dq_1}{dt} + B_1(\mu_1)\right) = \frac{\partial \mu_1}{\partial x}\left(\frac{dq_1}{dt} - y_1 + a_1 + a_2\right) \\ \frac{d\mu_2}{dt} = 0 = \frac{\partial \mu_2}{\partial x}\left(\frac{dq_2}{dt} + B_1(\mu_2)\right) = \frac{\partial \mu_2}{\partial x}\left(\frac{dq_2}{dt} - y_2 + a_1 + a_2\right)\end{aligned}\right\} \tag{2.19}$$

which coincides with the jump conditions for weak solutions

$$\left.\begin{aligned}\frac{dq_1}{dt} &= U(q_1) = (\mu_1 + \mu_2 - a_1 - a_2)\big|_{q_1} = \mu_2(q_1) - a_1 - a_2 = y_1 - a_1 - a_2 \\ \frac{dq_2}{dt} &= U(q_2) = (\mu_1 + \mu_2 - a_1 - a_2)\big|_{q_2} = \mu_1(q_2) - a_1 - a_2 = y_2 - a_1 - a_2 \,.\end{aligned}\right\} \tag{2.20}$$

Finally, differentiate y_1 and y_2 to find

$$\left.\begin{aligned}\frac{dy_1}{dt} &= \frac{\partial \mu_2}{\partial x}\frac{dq_1}{dt} + \frac{\partial \mu_2}{\partial t} = \frac{\partial \mu_2}{\partial x}\left(\frac{dq_1}{dt} + B_1(\mu_2)\right) \\ \frac{dy_2}{dt} &= \frac{\partial \mu_1}{\partial x}\frac{dq_2}{dt} + \frac{\partial \mu_1}{\partial t} = \frac{\partial \mu_1}{\partial x}\left(\frac{dq_2}{dt} + B_1(\mu_1)\right)\end{aligned}\right\}$$

and so

$$\left.\begin{aligned}\frac{dy_1}{dt} &= \mathrm{Sign}(y_1)(y_1 - a_1)(y_1 - a_2) \\ \frac{dy_2}{dt} &= \mathrm{Sign}(y_2)(y_2 - a_1)(y_2 - a_2) \,.\end{aligned}\right\} \tag{2.21}$$

Thus, the equations of evolution for y_1 and y_2 decouple from those of q_1 and q_2. The decoupled equations (2.21) can be solved first and q_1, q_2 subsequently determined from (2.20) by quadratures.

It is interesting to examine the connection of the set of variables q_i, y_i, $i = 1, 2$, with the q_i, p_i, $i = 1, 2$, introduced by Camassa and Holm [1993]. By definition, the q's are the same in both sets (the (q, y) system does have a canonical Hamiltonian form). As to the y's, notice that

$$\left.\begin{aligned} U(q_1) &= y_1 - a_1 - a_2 = p_1 + p_2 e^{-|q_1-q_2|} \\ U(q_2) &= y_2 - a_1 - a_2 = p_2 + p_1 e^{-|q_1-q_2|} \end{aligned}\right\} \tag{2.22}$$

which provides a definition of the transformation of the y variables in terms of p's and q's. This together with (2.20) yields the first set of equations for the Hamiltonian system derived by Camassa and Holm [1993],

$$\left.\begin{aligned} \frac{dq_1}{dt} &= p_1 + p_2 e^{-|q_1-q_2|} \\ \frac{dq_2}{dt} &= p_2 + p_1 e^{-|q_1-q_2|} \,. \end{aligned}\right\} \tag{2.23}$$

The constants of motion a_1 and a_2 can be expressed in terms of p's and q's via the first integrals

$$P_{12} = p_1 + p_2 = -(a_1 + a_2),$$

and

$$H_{12} = \frac{1}{2}(p_1^2 + p_2^2) + p_1 p_2 e^{-|q_1-q_2|} = \frac{1}{2}(a_1^2 + a_2^2)\,,$$

which are the total momentum and Hamiltonian for the (q, p)-flow, respectively. Using these expressions, the constants of motion a_1 and a_2 can be eliminated from (2.22) and the explicit variable transformation of variables for y_1, y_2 in terms of the (q, p) system is

$$\left.\begin{aligned} y_1 &= p_2(e^{-|q_1-q_2|} - 1) \\ y_2 &= p_1(e^{-|q_1-q_2|} - 1)\,. \end{aligned}\right\} \tag{2.24}$$

Differentiating these equations with respect to time, using (2.21), (2.20), and the transformation (2.24) itself, results in a system of equations for $\dot{p}_1$ and $\dot{p}_2$, which can be solved to yield

$$\left.\begin{aligned} \frac{dp_1}{dt} &= \operatorname{sgn}(q_2 - q_1) p_1 p_2 e^{-|q_1-q_2|} \\ \frac{dp_2}{dt} &= \operatorname{sgn}(q_1 - q_2) p_2 p_1 e^{-|q_1-q_2|} \,. \end{aligned}\right\}$$

Equations (2.23) and (2.24) are precisely those that follow from H_{12} with a canonical Hamiltonian structure in the (p, q) variables.

The case of three or more derivative-shock singularities x_i, y_i, $i = 1, 2, \ldots, N \geq 3$ proceeds in complete analogy with the case $N = 2$ above. Once again, the q-flows decouple from those of the y's, however the equations in the system governing the y-flow are now coupled and it is not immediately obvious that this system is integrable. A closer inspection however reveals that the y-flow shares the same structure as that of the μ-variables flow and is therefore integrable by a similar argument.

3 Generalized Peakons Lattices

In this section we discuss the connection between the peakons lattice and the Toda flows. The fact that the peakons lattice can be realized as a special case of the Toda flows in $gl(n)$ was discovered in Ragnisco and Bruschi [1996] and later used by Suris [1996] to introduce a discrete time peakons lattice. Let us review these results, concentrating, for simplicity, on a particular case of the peakons lattice with a Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{1\le i<j\le n} p_i p_j \exp(q_j - q_i) \ , \tag{3.1}$$

that corresponds to the case when particles q_i are ordered as follows: $q_1 > q_2 > \ldots > q_n$. It turns out that (3.1) possesses a Lax representation with the Lax matrix

$$L = L(p,q) = \sum_{i=1}^{n} p_i E_{ii} + \sum_{1\le i<j\le n} \sqrt{p_i p_j}\exp\left(\frac{1}{2}(q_j - q_i)\right)(E_{ij} + E_{ji}), \tag{3.2}$$

and an auxiliary matrix equal to the half of the skew-symmetric part of L, the peakons lattice thus being a restriction of the Toda flow to the set of matrices of the form (3.2). It was shown that this set forms a Poisson submanifold w.r.t. the associated r-matrix bracket on $gl(n)$. Complete integrability of (3.1) is provided by the Chevalley invariants of $gl(n)$, i.e. spectral invariants of L.

In what follows we give a uniform Lie-algebraic procedure which allows us to construct peakon-type orbits in any simple Lie algebra. On each of these orbits the Toda flow is shown to be a completely integrable system. Notice that for proving this we provide additional first integrals which supplement the Chevalley invariants of the algebra. We also indicate a convenient way of constructing Darboux coordinates for orbits under consideration.

Let us first recall the Hamiltonian formalism for the generalized (symmetric) Toda flows (cf. Kostant [1979], Goodman and Wallach [1985], Reyman and Semenov-Tian-Shansky [1994]). Let $\mathfrak{g}$ be the normal real form of a simple Lie algebra of rank r, $\mathbf{G}$ a corresponding Lie group and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by Φ the root system of $\mathfrak{g}$, and by Φ^+ (resp. Φ^-) the set of all positive (resp. negative) roots. We also fix a Chevalley basis $\{e_\alpha,\ \alpha\in\Phi;\ h_i,\ i=1,\ldots,r\}$ in $\mathfrak{g}$. All properties of the root systems and Chevalley bases that we need can be found in Humphreys [1980] and Onishchik and Vinberg [1990].

Consider a direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}_+$, where $\mathfrak{b}_+$ is the upper Borel subalgebra and $\mathfrak{k}$ is the maximal compact subalgebra of $\mathfrak{g}$. The dual space $\mathfrak{b}_+^*$ of $\mathfrak{b}_+$ can be identified with the space S of "symmetric" elements of $\mathfrak{g}$ that have a form

$$L = \sum_{j=1}^{r} b_j h_j + \sum_{\alpha\in\Phi} a_\alpha(e_\alpha + e_{-\alpha})\,. \tag{3.3}$$

S is an annihilator of $\mathfrak{k}$ w.r.t the Killing form $\langle\ ,\ \rangle$.

The pull-back of the Lie-Poisson bracket on $\mathfrak{b}_+^*$ equips S with the Poisson bracket

$$\{f_1, f_2\}_S(L) = \frac{1}{2}\langle L, [\pi_+\nabla f_1(L), \pi_+\nabla f_2(L)]\rangle \ , \tag{3.4}$$

where gradients are defined w.r.t. the Killing form and π_+ is a projection on $\mathfrak{b}_+$ parallel to $\mathfrak{k}$. Equations of motion of the generalized Toda flow on S are generated by a Hamiltonian $H(L) = \frac{1}{2}\langle L, L\rangle$ and have the Lax form

$$\dot{L} = [L, \frac{1}{2}\pi_+(L)] = [\frac{1}{2}\pi_-(L), L] \tag{3.5}$$

where $\pi_- = Id - \pi_+$. The bracket (3.4) is a restriction to S of the so called r-matrix bracket on $\mathfrak{g}$:

$$\{f_1, f_2\}_r(X) = \frac{1}{2}\langle X, [\pi_+\nabla f_1(X), \pi_+\nabla f_2(X)] - [\pi_-\nabla f_1(X), \pi_-\nabla f_2(X)]\rangle \ . \tag{3.6}$$

Symplectic leaves of the Poisson manifold $(S, \{,\}_S)$ coincide with the orbits of the coadjoint action of the upper Borel subgroup $\mathbf{B}_+$ of $\mathbf{G}$:

$$\mathrm{Ad}^*_b(L) = \pi_S \mathrm{Ad}_{b^{-1}}(L)\,, \tag{3.7}$$

where π_S is a projection on S parallel to $\mathfrak{b}_+$. On every indecomposable symplectic leaf, restrictions of the Chevalley invariants $I_1 = H, I_2, \ldots, I_r$ of $\mathfrak{g}$ form a family of independent Poisson commuting integrals for the Toda flow. This family is maximal, however, only for a few distinguished orbits, that were classified in Goodman and Wallach [1984] and Perelomov and Kamalin [1985]

To describe a particular type of coadjoint orbits, which in the $sl(n)$ case will be shown to be associated with the peakons lattice, we need the following definitions introduced in Gekhtman and Shapiro [1999]. Let m be the maximal positive root and

$$h_m = [e_m, e_{-m}] \ . \tag{3.8}$$

Define

$$\begin{gathered} \mathfrak{g}' = \mathrm{Ker}_{\mathrm{ad}_{e_m}} \cap \mathrm{Ker}_{\mathrm{ad}_{e_{-m}}} \\ F = \mathrm{Span}\{e_\alpha : \alpha \in \Phi^- \text{ and } (m, \alpha) \neq 0\} \\ \tilde{F} = \mathrm{Span}\{F, h_m\} \\ V = F - \mathbb{R}\langle e_{-m}\rangle \ . \end{gathered} \tag{3.9}$$

Then $\mathfrak{g}'$ is a semisimple subalgebra of $\mathfrak{g}$ and F is a Heisenberg subalgebra of $\mathfrak{g}$, i. e. V is spanned by root vectors $e_{\alpha_i}, e_{-m-\alpha_i},\ i = 1, \ldots N$, such that

$$\begin{gathered} [e_{\alpha_i}, e_{\alpha_j}] = [e_{-m-\alpha_i}, e_{-m-\alpha_j}] = [e_{-m}, e_{\alpha_i}] = [e_{-m}, e_{-m-\alpha_i}] = 0, \\ [e_{\alpha_i}, e_{-m-\alpha_j}] = c_i \delta_i^j e_{-m},\ , \end{gathered} \tag{3.10}$$

where c_i are some positive constants. Note also that

$$[h_m, e_{\alpha_i}] = -e_{\alpha_i},\ [h_m, e_{-m-\alpha_i}] = -e_{-m-\alpha_i},\ [h_m, e_{\pm m}] = \pm 2e_{\pm m} \ , \tag{3.11}$$

Therefore (3.4) and (3.11) yield the following Poisson brackets

$$\{y_0, x_0\}_S \ = x_0,\ \{y_0, x_i\}_S \ = \frac{1}{2}x_i,\ \{y_0, y_i\}_S \ = \frac{1}{2}y_i,\ \{y_i, x_i\}_S \ = \frac{1}{2}x_0,\ i = 1, \ldots N, \tag{3.12}$$

for the linear functions

$$x_i = \langle L, e_{-\alpha_i}\rangle,\ y_i = \frac{1}{c_i}\langle L, e_{m+\alpha_i}\rangle,\ x_0 = \langle L, e_m\rangle,\ y_0 = \langle L, h_m\rangle \ . \tag{3.13}$$

All other brackets between functions (3.13) are zero.

Let $h_0 \in \mathfrak{h}$ be orthogonal to h_m w.r.t the Killing form and let $\mathfrak{O}_{m,h_0}$ be the coadjoint orbit of $\mathbf{B}_+$ through $(e_{-m} + h_0 + e_m)$.

Theorem 3.1 *$\mathfrak{O}_{m,h_0}$ can be parameterized by elements of $\tilde{F}$:*

$$\mathfrak{O}_{m,h_0} = \left\{ L = \pi_S \left(\zeta - \frac{1}{2x(\zeta)} [ad_{e_m} v(\zeta), v(\zeta)] \right) \ : \zeta \in \tilde{F}_-, x(\zeta) > 0 \right\} \ , \tag{3.14}$$

where element $\zeta \in \tilde{F}_-$ is decomposed as

$$\zeta = x(\zeta)e_{-m} + y(\zeta)h_m + v(\zeta) \ ,$$

and $v(\zeta) \in V$.

The Toda flow (3.5) is completely integrable on $\mathfrak{O}_{m,h_0}$ with a maximal family of functions in involution given by the coefficients of polynomials

$$I_j \left(L + \frac{\lambda}{x(\zeta)} e_m \right) = \sum_k I_{jk}(L)\lambda^k, \ j = 1, \dots r, \tag{3.15}$$

where I_j are the Chevalley invariants of $\mathfrak{g}$.

The proof makes use of the so-called 1-chop map, a Poisson map from $\mathfrak{g}$ to $\mathfrak{g}'$, which was introduced for $sl(n)$ in Singer [1990, 1993] and generalized for an arbitrary Lie algebra in Gekhtman and Shapiro [1999]. This map and the functions (3.15) play a key role in the proof of the complete integrability of the Toda flows on generic orbits (see Deift *et al.* [1986] and Ercolani *et al.* [1993] for the $sl(n)$ case and Gekhtman and Shapiro [1999] for a general case).

It also follows from Theorem 2.1 that for constructing Darboux coordinates on $\mathfrak{O}_{m,h_0}$, it is sufficient to construct Darboux coordinates for a rather simple Poisson algebra generated by (3.12).

The following example demonstrates the connection with the peakons lattice.

Example 2.2. Let $\mathfrak{g} = sl(n)$. Then $e_m = E_{1n}, e_{-m} = E_{n1}, h_m = E_{11} - E_{nn}$,

$$\tilde{F} = \left\{ \zeta = \begin{bmatrix} y & 0 & 0 \\ v_1 & 0 & 0 \\ x & v_2^T & -y \end{bmatrix} \ : \ x, y \in \mathbb{R}, v_1, v_2 \in \mathbb{R}^{n-2} \right\} \ .$$

Since $\mathrm{Tr}(L)$ is invariant under the action (3.7), we can define S to be the space of all symmetric matrices, not necessarily with zero trace. One can choose h_0 to be any diagonal matrix of the form $h_0 = \mathrm{diag}(\kappa, d, \kappa)$, where d is a $(n-2) \times (n-2)$ diagonal matrix. It follows from (3.14) that

$$\mathfrak{O}_{m,h_0} = \left\{ L = \begin{bmatrix} y + \kappa & v_1^T & x \\ v_1 & d + \pi_S(x^{-1} v_1 v_2^T) & v_2 \\ x & v_2^T & \kappa - y \end{bmatrix} \ : \ x > 0, y \in \mathbb{R}, v_1, v_2 \in \mathbb{R}^{n-2} \right\} \ . \tag{3.16}$$

If $d = 0$, an open set $\{-\kappa < y < \kappa; \ v_{1i} > 0, \ v_{2i} > 0, \ i = 1, \dots, n-2\} \subset \mathfrak{O}_{m,h_0}$ admits Darboux coordinates p_i and q_i defined as follows:

$$p_1 = y + \kappa, \ p_n = \kappa - y, \ p_{i+1} = \frac{v_{1i} v_{2i}}{x} \ , \ i = 1, \dots, n-2,$$

and

$$\exp(q_n - q_1) = \frac{x^2}{\kappa^2 - y^2}, \ \exp(q_i - q_1) = \frac{x v_{1i}}{(y + \kappa) v_{2i}} \ , i = 2, \dots, n-1.$$

With this parameterization matrix L in (3.16) coincides with the Lax matrix (3.2). Note that $p_1 + \cdots + p_n = const$ on $\mathfrak{O}_{m,h_0}$ and that coordinates q_i are defined only up to a translation. Note that a different choice of Darboux coordinates for $\mathfrak{O}_{m,0}$ was proposed in Kamalin and Perelomov [1985] as an example of the general construction of canonical coordinates on coadjoint orbits.

Example 2.3. Let $\mathfrak{g}$ be an exceptional algebra of type G_2. Denote the short and long simple roots of $\mathfrak{g}$ by ν_1, ν_2 resp., and root vector corresponding to a negative root $\alpha = -(i\nu_1 + j\nu_2)$ by e_{ij}. Then F is spanned by vectors $e_{-m} = e_{32}$ and $e_{i1},\ i = 1, \dots 4$. Thus, for any h_0 the orbit $\mathfrak{O}_{m,h_0}$ is 6-dimensional, while there are only two independent Chevalley invariants. This illustrates the necessity of using functions (3.15) to establish complete integrability.

Put $h_0 = 0$ and choose Darboux coordinates q_i, p_i for Poisson algebra (3.12) with $N = 3$ in the form:

$$q_1 = x_0,\ p_1 = -\frac{y_0}{x_0},\ q_2 = -\frac{x_1}{3\sqrt{x_0}},\ p_2 = \frac{6y_1}{\sqrt{x_0}},\ q_3 = -\frac{y_2}{\sqrt{x_0}},\ p_3 = -\frac{2x_2}{\sqrt{x_0}}\ .$$

Then Theorem 2.1 gives the following integrable polynomial Hamiltonian quadratic in momenta:

$$H(p,q) = \frac{1}{12}(p_3q_2 + 6q_3^2)^2 + \frac{1}{8}(p_3q_3 + p_2q_2)^2 + \frac{q_1}{12}(p_3^2 + 12p_2^2 + 3p_1^2) + q_1^2 + 3q_3^2q_1 + 3q_3^4 \qquad (3.17)$$

A detailed analysis of the system generated by (3.17), as well as integrable systems associated with peakon-type orbits in other simple Lie algebras will be given in a forthcoming publication, Alber and Gekhtman [1999].

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